

Original Research Article

How Many Public and Private Schools We Need

ABSTRACT

This paper demonstrates that rationality is closely related to knowledge, and personal attainment is positively related with received education, and social development is based on the corresponding knowledge and education. We analyzed the different functions of private and public education, the price formation of public education, and the quantity of public and private schools. We demonstrated the importance of education market and university ranking in the development of education, and analyzed the proportion of people engaged in education.

JEL Codes: H00, I21

Key words: knowledge; development; education; school.

1. INTRODUCTION

Natural science makes people know more and deeper about the universe, and technology enables people make new products, explore more and more deep into the outer space, and social science enable people establish cost-efficient economic and political institutions. Education aims to impart knowledge to students, enable them understand where we come, where we will go and what rules we need to obey.

Usually people are supposed to be rational, which means people will maximize their utilities or profit, but this may not be true if people do not have adequate knowledge. For example, a child cannot maximize his long-run utility before he grows up, and a young man cannot operate a complex machine without professional training.

In the competition for survival, the people with more knowledge and physical power will win, and education aims to impart knowledge to people, enables people master necessary knowledge and skills, making people more rational. So rationality is positively related with knowledge. Homogeneity will be high if education gaps among people are small, and heterogeneity will be high if education gaps among people are great. A market may be perfect if all economic agents are at the same level of education. The relationship of rationality with knowledge is indicated in figure 1, and the relationship of market efficiency with knowledge gap is indicated in figure 2.

Then, a question arises, where does knowledge come from, and who will take research of the urgent problems that people are eagerly confronting with. In early societies, it is old people who have accumulated valuable experience and impart them to the youth, such as skills of making instruments, hunting, planting and so on. Gradually some experts established schools teaching students.

Early formal schools were documented in ancient Greece, China and India, though nobody knows exactly when the first school was created. Xiong [1] indicated that there are government-established schools in the West Zhou Dynasty in China, schools are affiliated to and controlled by governments, names, ranks of schools affiliated to the central and local governments are different (1046-771 B.C.). Yang [2] indicated that Confucious is one of the first people who established private schools in Spring and Autumn Period in China (about 520 B.C.). Table 1 indicates the top 10 oldest schools in the world.

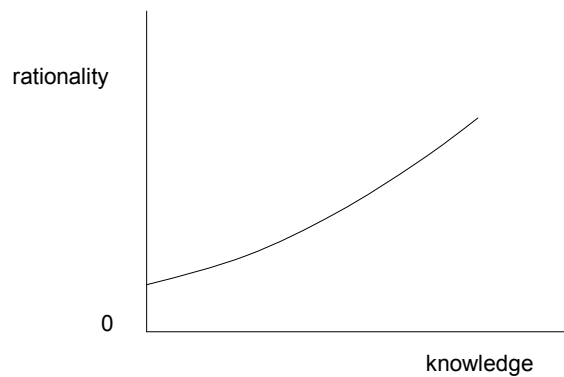


Figure 1: The relationship of rationality and knowledge

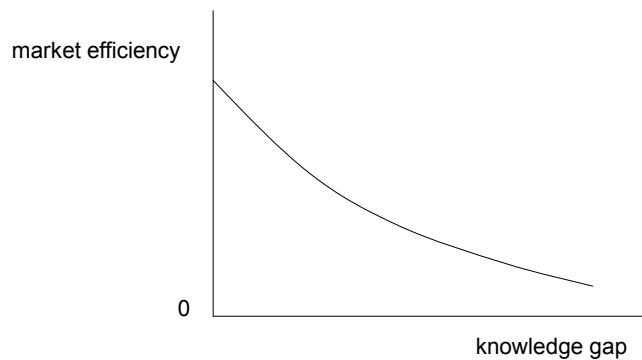


Figure 2: The relationship of market efficiency and knowledge gap

Table1: The oldest schools in the world: top 10

name	location	time	type	country
Chengdu Shishi High School	Chengdu	143–141 B.C.	Public school	China
The King's School,	Canterbury,	597 A.D	Private school	England
The King's School,	Rochester	604 A.D.	Private school	England
St Peter's School	York	627 A.D	Private school	England
Thetford Grammar School	Thetford	631 A.D.	Private school	England
Hereford Cathedral School	Hereford	676 A.D.	Private school	England
Royal Grammar School Worcester	Worcester	685 A.D	Private school	England
Beverley Grammar School,	Beverley	700 A.D.	state school	England
Gymnasium Paulinum	Münster	797 A. D	a cathedral school	Germany
Gymnasium Theodorianum	Paderborn	799 A.D.	Public school	Germany

Resource: Wikipedia, List of the oldest schools in the world [3].

Some scholars oppose conventional styles of schooling, indicated that the conventional schooling would engulf the life time of a student, educational expenditure was a burden to students, and suggested that de-institutionalizing education may be a starting point for a de-institutionalized society [4]. Hern [5] indicated that there is no right way to educate a child, the conventional education deadens children's natural love for learning, undercuts their self-esteem, and limits independent thought. He suggested replacing compulsory schooling with a wide variety of home, neighborhood and community-based education.

The conventional education system has shortcomings, and reforms have been made to meet the demands of students. A school is a place where experts conduct research and teach students. A school is also an authority and a third party that assures us that a student who has received a diploma is well educated, and school is the symbol of institutional progress. School

98 itself is not a problem, and the problem is what kind of school we need. We need to trade off
99 the benefit and cost of schools, we can reduce schools to a necessary quantity, but adequate
100 schools will do benefit to our society.

101
102 The outline of the study is organized as follows: Section 1 is introduction. The relationship of
103 career attainments and education is analyzed in section 2. The relationship of social
104 development with knowledge and education is analyzed in section 3. Different functions of
105 private and public education are analyzed in section 4. In section 5, price formation of public
106 education and the quantity of public and private schools are analyzed, and we emphasized the
107 importance of education market and university ranking in the development of education. We
108 present a model analyzing the proportion of people engaged in education in section 6. Section
109 7 is discussion and conclusion.

110 111 **2. CAREER ATTAINMENTS AND EDUCATION**

112 113 **2.1. Entrepreneurship and Education**

114
115 Advanced business education can increase the propensity of people toward entrepreneurship,
116 formal education can increase entrepreneurial efficiency and successful firm growth, the
117 number of years of formal education increase the probability of becoming self-employed [6].
118 On-the-job training may be more important than the quality of the training [7], expertise in
119 business is fundamental to the survival and success of small business [8].

120
121 Personality traits are partly developed by innate nurturing, socialization and education [9].
122 Education system have a crucial role in developing the entrepreneurial traits of students, for
123 example, curricula are very important in the developing of independence, creativity and risk
124 taking [10].

125
126 Some socialists [11] indicated that the influence of education on being an entrepreneur is not
127 linear. Black and Smith [12] found that the effect of quality of college on returns to education
128 diminished as other factors were included, such as ability and demographics, and they find
129 that attending an elite university increase the wages of men by 11 percent and women by 7.5
130 percent.

131
132 Loury and Gardman [13] found that selectivity did not play a very important role on the returns
133 to education. Goedhuys and Sleuwaegen [14] indicated that primary education does not have a
134 significant impact on being an entrepreneur, but higher education has a significant impact on
135 being an entrepreneur.

136
137 A higher level of educational attainment does not necessarily increase risk taking propensity of
138 students, but a higher level of educational attainment can provide an individual with more
139 confidence to become entrepreneurially active, a student with a higher level of education has a
140 higher intention to become an entrepreneur [15].

141
142 Wu and Wu [16] reported that respondents with a diploma and undergraduate degree show
143 higher interest in starting a business than those with a post graduate degree. According to the
144 researchers, individuals with higher levels of education may have many options, which
145 decrease the intention to entrepreneurship of students [15].

146 147 **2.2. Career Attainments and Education**

148
149 Education had a positive effect on professionalism, overall performance and organizational
150 commitment [17]. Education has a positive effect on career development [18], and elites tend to
151 have more higher education in developed countries [19]. A master's degree has a larger
152 positive effect on promotion probability and on the rate of salary increases [20]. Managers with
153 a bachelor's degree earned substantially less than managers with a master's degree [21].
154 Degrees above masters have also been found to add significantly to future earnings, career
155 attainment will be positively related to number of years of schooling [22].

156
157 Many corporate leaders come from a few top ranked schools, graduating from an Ivy League

school can increase a manager' salary [21]. The more selective a school, the more likely a student will receive a high salary [23]. Attending an elite private institution will generate increased economic returns [24]. Baruch and Leeming [25] indicated that return from educational attainment is significant, such as salary, promotion, satisfaction level with a job.

In July 2002, the US Census Bureau indicated that a high school graduate can get an average salary of USD 18.900, a college graduate can get an average salary of USD 45.400, and a graduate with a professional degree can get an average salary of USD 99.300 [18]. The rates of return to education vary overtime, but education is becoming more and more important in career attainment in recent years [26].

In OECD countries, people received tertiary education have the highest employment rate. The unemployment rate of people received an upper secondary education is 13%, and unemployment rate of people received the tertiary education is 5% [27].

2.3. Curriculum, Social Culture and Career Attainments

Curriculum is important factor that influences attainment. Loury and Gardman [13] found that college majors had a significantly large effect on earnings. Larkins [28] found that a bachelors' degree of science and technology yield greater returns than humanities and social science. College majors associated with greater career attainment are engineering and business [29]. Spileman and Lunde [20] found that the probability with an engineering major of being promoted was high, and it is positively associated with promotion rate.

Career attainment is also affected by other variables, such as campus culture, family culture, economic condition and geographic location [24]. School education may not be the direct cause of salary differentials, but employers are interested in hiring intelligent individuals [30]. Effect of quality of college on returns to education was influenced by many individual factors, such as ability and demographics [12].

3. KNOWLEDGE, EDUCATION AND SOCIAL DEVELOPMENT

Generally, education is a form of learning, in which the knowledge, skills, and habits of a group of people are transferred from one generation to the next through teaching, training, or research [31]. World scientific culture has provided a strong support for education expansion, and worldly wide competition promotes economic development of nation countries [32, 33]. Goodlad and McMannon [34] indicated that the youths must be enculturated to comply with social and political expectations, such as civility and Civitas.

Weber [35] indicated that the origins of industrial revolution can be traced to the Protestant reformation and the rise of Calvinism. Different religions and culture are key factors that determine the economic performance and equilibrium outcomes for a given set of institutions, such as beliefs, world views, values and preferences of a society [36].

Education is viewed as a most efficacious instrument for modernization by developing countries [37]. The United Nations Educational, Scientific and Cultural Organization (UNESCO) was established in 1945 to promote intercultural understanding, long lasting peace and development, UNESCO takes education as a fundamental human right and a prerequisite for human development, and every child has the right to quality education. Economic and political agreement may not adequately guarantee the long lasting order and peace, and order and peace must be established on the basis of humanity's moral and intellectual solidarity, which is one of the aims of the UNESCO.

3.1. Productivity and Education

Human capital is a necessity to the social development, such as the training of general education, firm-specific training and relevant experience [21]. Schultz [38] introduced human capital into the production function to resolve the residual puzzle to explain economic growth. Education can enhance productivity and make an individual earn more money, so education is

217 a major impetus to development [39].

218
219 Education can promote economic growth, its influence may not be prompt, but it will show its
220 effect at later times. The study of Peaslee [40] suggests that there may be a lag of 8 years for
221 the tertiary education to play a role, and a lag of 12 years for the secondary education. But the
222 lag may be short in present days because of the information technology such as internet.

223
224 A certain quantitative education is requisite for economic development, but education alone
225 does not assure economic development, other economic factors, behavioral factors, political
226 factors exert varying influences to economic development [40]. It is increasingly recognized
227 that knowledge and innovation are key to sustaining economic growth, high level of skills and
228 knowledge can create and fuel knowledge based economies.

229
230 An average of 83% of today's young people in OECD countries will complete upper secondary
231 education over their lifetimes. 79% of young people will complete upper secondary education
232 over their lifetimes in G20 countries. On average, 60% of young adults (67% of women and 53%
233 of men) in OECD countries are expected to enter tertiary-type A programme over their life,¹ 3%
234 are expected to enter advanced research programmes, 19% of today's young adults (20% of
235 women and 18% of men) will enter tertiary-type B programmes over their lifetimes [27].²

236 237 **3.2. Human Capital and Education**

238
239 It is acknowledged conventionally that education is a central ingredient of economic
240 development, and can produce valuable economic returns [32]. Smith [41] indicated that
241 education can promote productive efficiency, as well as individual and social development,
242 Walsh [42] took education as an investment because the physical input including land,
243 machines and labors cannot account for all the output.

244
245 Educational (or manpower) planning is a central ingredient of economic development. Inkeles
246 [43] indicated that education is directly related with modernization in six developing countries.
247 Education is more effective than other factors on modernization, such as work experience or
248 urbanization [44, 45]. The study of Peaslee [40] demonstrates that there is a close relationship
249 between economy and high primary education enrollments. The relationship between illiteracy
250 and poverty, hunger and disease throughout the world is accepted increasingly [46].

251
252 Education is a social investment, and can produce valuable economic returns [32]. Private
253 returns on investment in tertiary education are comparatively substantial, it pays off the
254 expenditure of an individual, and the public also benefit in the form of tax revenues and social
255 contributions. The net public return of tertiary education of a man is over USD 100 000 across
256 OECD countries, it is about three times the amount of public investment in education. The
257 public return of a women is around USD 60 000, it is about twice the amount of public
258 investment [27].

259 260 **3.3. Economic, Political Institutions and Education**

261
262 Education create and legitimate modern citizen and elite roles in the political system, and
263 emphasize the responsibility of citizenry, political development and education are reciprocally
264 related [47]. Modern societies are schooled, based on the allocation of authority of education,
265 and allocate political rights and capacities to people with appropriate credentials [4].

266
267 National educational systems have expanded rapidly after the Second World War [48],
268 Contemporary educational systems are politically constructed institutions, and they are
269 structured by political authorities to incorporate human resources. The federal government of

¹ Programmes designed to provide qualifications for entry to advanced research, such as medicine, dentistry or architecture. Duration is at least 3 years full-time. Tertiary-type A programmes include second-degree programmes, such as the American master's degree.

² Programmes are typically shorter than those of tertiary-type A and focus on practical, technical or occupational skills for direct entry into the labor market, and they have a minimum duration of two years full-time equivalent at the tertiary level.

USA passed the first federal statutes and established the first federal office of education during and after the Civil War, it is seen as a way to consolidate its position in the South [49, 50]. Communist states and other totalitarian states control higher education to form and legitimate the political center and authority [51, 52].

We know that some basic rules have to be implemented to insure the cooperation of the society, such as ethics, norms and laws. And a third party is important to implement the rules and punishment in case that some person violates the rules. A third party may be a court, a department of the government that implements the rules and punishment [53]. Laws and third parties are important components of economic and political institutions, and knowledge and education promote the establishment of the institutions. Figure 3 indicates the development of culture and institutions along with that of knowledge and education.

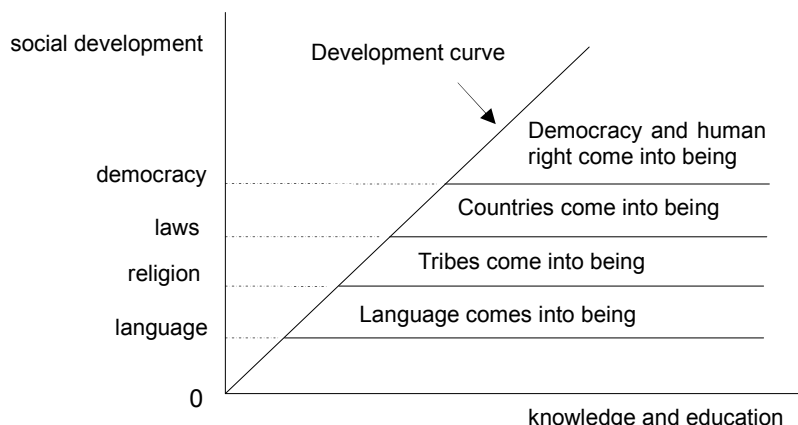


Figure 3: The Relationship of Social development with knowledge and education

4. DIFFERENT FUNCTION OF PRIVATE AND PUBLIC EDUCATION

Public goods are non-excludable and non-subtractable [54], or non-excludable and non-rival [55]. Obviously, knowledge has the two characteristics of public goods, and we take it as public good. Knowledge makes education an impure public good although education can be private managed [56]. Benjamin R. Barber [34] indicated that schools are public because they serve public purposes, and multicultural education is essential to maintain democracy. Of course, linguistic pluralism may be an impediment to universal literacy [46].

McCulloch indicated that a better law of inheritance and a better system of education are two powerful instruments to reduce inequality of income [57]. Public education reduces income inequality more quickly than private education when social income inequality is large, and private education is more prone to higher per capita income [58].

Private schooling leads to higher growth when the gap of human capital endowments across families is small, and public education has higher growth when the gap of human capital is big. Public schooling leads to income convergence, while private schooling results in ever increasing inequality [59]. So public education plays a role of redistribution of social income, and the mount of public schools should increase when income gap increases, and it should decrease when income gap decreases.

4.1. Private Education can enlarge income gap

We know that investment in education of a child is a question of wealth distribution between two generations. Suppose that there are two persons in a family, a father and a son, they have a two-period problem: in the first period, the father has to make a decision whether he will let the boy go to school. The boy will get a normal job in the second period if he is well educated in the first period; he will get a low income job in the second period if he is not well educated or not educated at all. A normal job will be paid a normal salary of b regardless of the periods.

The boy will get a salary of b if he gets a normal job in the second period, and he will get a salary of $\frac{b}{n}$ if he gets a low income job, $n=1, 2, \dots$. Suppose the probability that the boy is well educated is p , the cost of education is c , the rate of bank interest is r , so we have an equation:

$$p(b-c(1+r))+(1-p)(\frac{b}{n}-c(1+r)) = \frac{b}{n} \quad (1)$$

$$\Rightarrow c = \frac{bp(1-\frac{1}{n})}{1+r}$$

We can see that c is positively related with the normal income b , the probability p , and the income gap $(1-\frac{1}{n})$, and negatively related with the rate of bank interest.

Suppose at time t , there is a proportion α_0 of families that have an income lower than c , they cannot afford the tuition fee. Then the proportion of families that can afford tuition fees is $1-\alpha_0$. We have supposed that the probability a boy get well educated is p , then at time $t+1$, the proportion of children who are not well educated is $(1-\alpha_0)(1-p)$. The not educated plus not well educated is $\alpha_0 + (1-\alpha_0)(1-p)$, these children will have low income jobs. Suppose a small proportion β of those children who have low income jobs at time t can get normal jobs at time $t+1$, then the proportion of children who have low income jobs at time $t+1$ will be $\alpha_0 + (1-\alpha_0)(1-p) - \alpha_0\beta$, and it is $(1-\beta)\alpha_0 + (1-\alpha_0)(1-p)$. Then we have the recursion formula :

$$a_{n+1} = (1-\beta)a_n + (1-\alpha_0)(1-p) \quad (2)$$

$$\Rightarrow a_{n+1} = (p-\beta)a_n + 1-p$$

$$a_{n+1} = \begin{cases} \text{increase, if } p > \beta \\ 1-p, \text{ if } p = \beta \\ \text{decrease if } p < \beta \end{cases}$$

The series will be convergent if $p \leq \beta$, the series will be divergent if $p > \beta$, which means the proportion of children who have a low income jobs will increase. Generally in an undeveloped economy, p is larger than β , the children who have low income jobs will increase.

Suppose a_n has been the biggest, and $a_{n+1} = a_n$, and from $a_{n+1} = (p-\beta)a_n + 1-p$, we can get $a_n = \frac{1-p}{1-p+\beta}$, so the maximum value is $\frac{1-p}{1-p+\beta}$, the proportion of children with low income jobs is included in the region: $(\alpha_0, \frac{1-p}{1-p+\beta})$.

$$\frac{1-p}{1-p+\beta} = \begin{cases} 0 & \text{if } p = 1 \\ 1 & \text{if } p = 0 \text{ (and } \beta = 0) \\ (0, 1) & \text{if } 0 < p < 1 \end{cases}$$

So we have a conclusion: (1) at the equilibrium point, the proportion of children with low income jobs is determined by p and β , it is not determined by the first value α_0 ; and (2) the proportion of children with low income jobs will approach 0 if p approaches 1, and the proportion will approach 1 if p approach 0 (under the condition $\beta = 0$).

4.2. The Role of Public Education

Compulsory education plays a role of redistributing social resources, enhance welfare and make distribution more even in a society [60]. Minimum schooling laws will decrease the inequality of income distribution [61], and the majority of individuals will be better off under a certain level of compulsory education in the long run. Theodore Sizer indicated that public education will give every child intellectual and civic tools, and enable children to have decent lives in modern culture and economy [34].

Public education is a way to counteract or weaken the mechanism of private education that enlarges income gap. Private education can enlarge income gap, and we can lessen the income gap by providing public education to children from poor families. Free access to basic

public education may provide the only chance to overcome poverty when parental human capital is low. By contrast, at advanced stages of development, free public education crowds out private education investment if parental human capital is high, and stimulates fertility and may impede growth [62].

In England, compulsory education was taken in 1870 by the Elementary Education Act [63], Frederic the Great of Prussia began compulsory elementary education in 1763, and Maria Theresa of Austria began compulsory elementary education in 1773 [64]. Having children in school for a specified period was adopted in the eighteenth century, and were followed by other countries, the majority of countries adopted compulsory basic education after the Second World War [65]. In the 20th century, compulsory basic education is unanimously accepted as an essential basic public service, and the compulsory education law is widely observed and enforced in all countries [64].

5. QUANTITY OF PUBLIC AND PRIVATE SCHOOLS

5.1. Price of Public Education

Knowledge is public good, and it has the characteristic of scale economy and strong external effect, this enables education have the characteristic of strong external effect and scale economy. Scale economy means marginal cost will decrease along with the increase of output, and the per capita education price will be lower if one person (or corporation) manages the supply of public education than two persons, but one person means monopoly, and the market power of monopoly will lift the price. To take the advantage of scale economy, and avoid the disadvantage of monopoly, the managing of public education should be publicly bid.

Technically, within the design capacity, the marginal cost of education will decline, and the per capita education price will decline. Beyond the design capacity, marginal cost will increase, and the effect of scale economy will disappear.

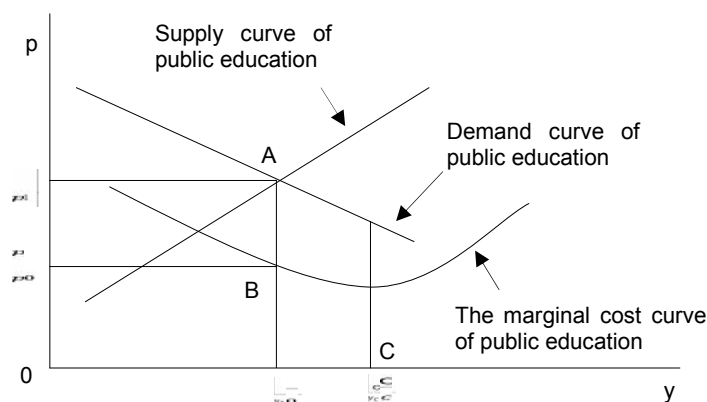


Figure 4: Price formation of public education

Generally the quantity of pupils who need the public education is approximately given, and different education managers have different marginal cost and bidding price. Figure 4 indicates the price formation of public education, y_0 is the total demand of education, it is approximately given. p_1 is the winning bid price, p_0 is the marginal cost. y_C is the design capacity, beyond point C, the marginal cost will increase.

So the price of public education is influenced by two factors, one is the quantity of managers who compete to manage public education, and the other is the quantity of pupils who need public education.

5.2. Quantity of Public and Private Schools

Suppose the combination of public schools and private schools is optimal, the output of the combination will cultivate the most qualified students. Suppose public schools are not profit making, and private schools are profit making. Tuition fees of public schools are low (or no tuition fees), tuition fees of private schools are high, teachers and education resources can flow freely at the education market.

Suppose the combination of public schools and private schools is like a Cobb-Douglas function, public school and private school are two factors, the total investment of public schools and private schools is a constant, and we want the optimum quantities of public schools and private schools. If the total investment is m , then we have the following optimum question:

$$\begin{aligned} \text{Max } (Y) \\ \text{s. t. } Y &= A x_1^\alpha x_2^\beta \\ p_1 x_1 + p_2 x_2 &= m \end{aligned} \quad \begin{aligned} (3) \\ (4) \end{aligned}$$

x_1 is the quantity of public schools, and x_2 is the quantity of private schools, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, p_1 is the tuition fee of public schools, p_2 is the tuition fee of private schools, the total investment of education (or endowment) is m . We have the answers of x_1 and x_2 :

$$x_1 = \frac{\alpha m}{(\alpha + \beta) p_1}, \quad x_2 = \frac{\beta m}{(\alpha + \beta) p_2}$$

Suppose $\alpha + \beta = 1$, we have: $x_1 = \frac{\alpha m}{p_1}$, $x_2 = \frac{\beta m}{p_2}$

We can see that the quantities of public and private schools are positively related with their income in the economy, and negatively related with the tuition fee.

Suppose that the public schools aim to meet the basic demand of education, they cannot enlarge or shut down in the short run. In the contrary, private schools will meet a variety of demands of the education market, a new school may be established, and an old school may be shut down. Suppose the growth rate of private schools is a , $1 \leq a \leq -1$, the added education income is $p_2 a x_2$, and the budget of total education is extended to $m + p_2 a x_2$, and the question is:

$$\begin{aligned} \text{Max } (Y) \\ \text{s. t. } Y &= A x_1^\alpha x_2^\beta \\ p_1 x_1 + p_2 x_2 &= m + p_2 a x_2 \end{aligned}$$

We have: $x_1 = \frac{\alpha m}{(\alpha + \beta) p_1}$, $x_2 = \frac{\beta m}{(\alpha + \beta)(1 - a) p_2}$. Suppose $\alpha + \beta = 1$, we have: $x_1 = \frac{\alpha m}{p_1}$, $x_2 = \frac{\beta m}{(1 - a) p_2}$.

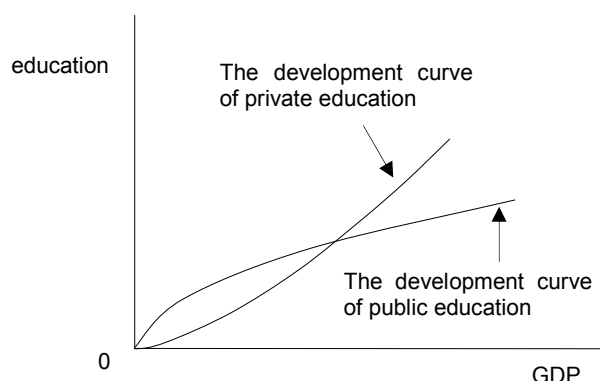


Figure 5: Development curves of public and private education

In a poor economy, people do not have much resource to invest in private education, and public schools may be the main way to get science and technology. With the development of economy, people have more resource to invest in private education, and different kinds of private schools can be built to satisfy different demands of people. The development curves of private and public education is indicated in figure 5.

Public education has the role of redistribution of income, so it should be positively related with income gap. Figure 6 indicates the relationship of public education with income gap.

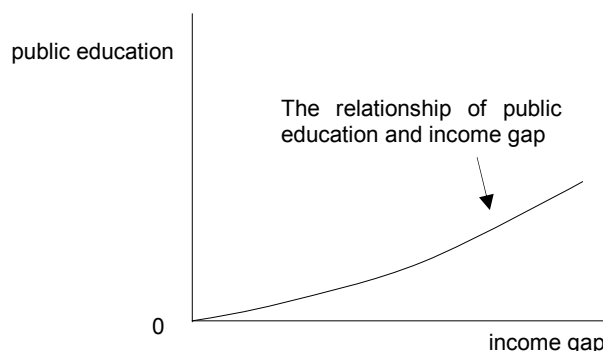


Figure 6: The relationship of public education and income gap

In most countries, public schools provide education from primary education to tertiary education. On average across OECD countries in 2011, about 89% of primary students, 86% of lower secondary students and 81% of upper secondary students are enrolled in public schools. Slightly less than 3% of primary students are enrolled in fully private schools, slightly more than 3% of lower secondary and more than 5% of upper secondary are enrolled in private schools. 15% of students of tertiary type A and advanced research programmes are enrolled in private schools, and the proportion will be 29% if government dependent private schools are included. 20% of students of tertiary type B programmes are enrolled in private schools, and the proportion will be 41% if government dependent private schools are included [27].

The United Kingdom is the only country that 100% of students of tertiary-type A and advanced research programmes are enrolled in government dependent private schools (or institutions), so do the students of tertiary-type B programmes [27].

5.3. Education Market and University Ranking

Chubb and Moe [66] indicated that public schools are bureaucratic, and they are governed by institutions of direct democratic control, institutional inertia may prevent effective organizational adaptation to changing environmental conditions [67]. Peterson [68] indicated that public schools made a noncompetitive environment and monopoly position, and it seemed that public schools were in a flourish.

The study of Coleman, Hoffer, and Kilgore [69] demonstrates that Catholic and other private schools have higher achievement than public schools, and academic, disciplinary policies, academic levels of students and their families can account for this. Jensen [70] indicated that higher level student academic performance in Catholic schools was mainly caused by the greater discipline and more rigorous requirements.

Majority voting will determine the education quality of schools, technically, the quality of a school is determined by the marginal return and the marginal cost to raise the education standard, the quality will be improved if the marginal return is greater than the marginal cost to raise education standard of the school. Such as USA and Italy, the USA produces a system with high quality of private schools and low quality of public schools, while Italy produce a system with high quality of public schools and low quality of private schools [71].

The QS (Quacquarelli Symonds) World University Rankings ranks universities annually, it

provide rankings according to academic reputation, employer reputation, faculty student, international faculty, international students and citations per faculty. The latest ranking in 2013 of the world's top universities has been released by the QS. The first is Massachusetts Institute of Technology (MIT), continues to rank first as last year, the second is Harvard University, the third is The University of Cambridge. Table 2 presents the world top 10 universities, the top 10 are dominated by universities of the U.S.A. and the United Kingdom.

Table 2: The top 10 universities in the world 2013

Rank	University	Type	Country
1	Massachusetts Institute of Technology	private research university	U.S.A
2	Harvard University	private research university	U.S.A
3	University of Cambridge	public collegiate research university (the colleges are private)	United Kingdom
4	UCL (University College London)	public research university	United Kingdom
5	Imperial College London	public research university	United Kingdom
6	University of Oxford	collegiate research university	United Kingdom
7	Stanford University	private research university	U.S.A
8	Yale University	private research university	U.S.A
9	University of Chicago	private research university	U.S.A
10	California Institute of Technology	private research university	U.S.A
10	Princeton University	private research university	U.S.A

Source: QS TOP UNIVERSITIES, QS World University Rankings 2013 [72].

Market competition is the force that furthers the development of education, an open, free, and fair market is a necessary platform that develops education. Obviously, university ranking is a good way to encourage schools to engage in education and research, just like the Nobel Prize being a way to encourage experts to achieve great academic success.

6. PROPORTION OF PEOPLE ENGAGED IN EDUCATION

6.1. Proportion of People Engaged in Education

Towns will be full of discontent students, urban slums and delinquents increase when students from primary school are more than jobs [51], people worried about over education, and schools may be in excessive supply. Universal education on the style of present schools will engulf the lifetime of students, it may not be feasible any longer, and educational webs may be alternate ways [4].

Chevalier [73] indicated that investment in education will cease at the point that the current value of future income equals to the cost of education. In a market economy, how many schools are needed is determined by the demand of education market, and education prices will solve the problem. Here we want to know the proportion of people who engaged in education, and the growing trend.

Suppose there are two sectors in an economy, one is labor, the other is education. Labor provides labor force, education provides knowledge. The optimum combination of them will produce the maximum output. Suppose the combination is like a Cobb-Douglas function, then we have the following function.

$$Y_t = A E_t^\alpha r_t^\beta L_t^\beta \quad (5)$$

We take E as education, which means knowledge comes from education, and L as labor. Y_t is the output at time t , E_t is the education at time t , r_t is the proportion of people engaged in labor at time t , L_t is the total available labor force of an economy at time t . $A > 0$, and A is a coefficient. Suppose E_t changes according to the following function.

$$\dot{E}_t + \delta E_t = B L_t (1 - r_t) \quad (6)$$

$\dot{E}_t$ is the change of education, it is an abbreviation of $\frac{dE_t}{dt}$, δ is a discount factor, it means that old knowledge may depreciate if new knowledge comes into being. ρ is a discount factor, it is the rate that we discount future income to present value. $B > 0$, and B is a coefficient, just like A . The question is indicated in the following functions.

$$\begin{aligned} \text{Max } & \sum_{t=0}^T e^{-\rho t} \ln(Y_t) \\ \text{s. t. } & Y_t = A E_t^\alpha r_t^\beta L_t^\beta \\ & \dot{E}_t + \delta E_t = B L_t (1 - r_t) \end{aligned}$$

And we have the answers of r_t and $1 - r_t$:

$$r_t = \frac{(\delta + \rho)\beta}{\alpha\delta + (\delta + \rho)\beta}, \quad 1 - r_t = \frac{\alpha\delta}{\alpha\delta + (\delta + \rho)\beta}$$

The proportion of people who engage in education is $1 - r_t = \frac{\alpha\delta}{\alpha\delta + (\delta + \rho)\beta}$, the procedure of solving the function is in appendix I.

We can see that the proportion of people engaged in education is getting bigger and bigger, α is getting bigger and bigger, if knowledge is becoming more and more important.

6.2. Proportion of People Engaged in Research and Education

Suppose research produce knowledge, and education imparts knowledge. There are three sectors in an economy, and they are research, education and labor. The optimum combination of them will produce the maximum output. Suppose the combination is like a Cobb-Douglas function, then we have the following function.

$$Y_t = A R_t^\alpha E_t^\beta r_{0t}^\gamma r_{1t}^\gamma L_t^\gamma \quad (7)$$

And education and research changes according to the following functions respectively:

$$\dot{E}_t + \delta_1 E_t = B_1 L_t r_{1t} \quad (8)$$

$$\dot{R}_t + \delta_2 R_t = B_2 L_t (1 - r_{0t} - r_{1t}) \quad (9)$$

The dynamic optimization question is:

$$\begin{aligned} \text{Max } & \sum_{t=0}^T e^{-\rho t} \ln(Y_t) \\ \text{s. t. } & Y_t = A R_t^\alpha E_t^\beta r_{0t}^\gamma r_{1t}^\gamma L_t^\gamma \\ & \dot{E}_t + \delta_1 E_t = B_1 L_t r_{1t} \\ & \dot{R}_t + \delta_2 R_t = B_2 L_t (1 - r_{0t} - r_{1t}) \end{aligned}$$

R_t is the research at time t , E_t is the education at time t , L_t is the labor at time t . $\dot{E}_t$ is the change of education at time t , it is the abbreviation of $\frac{dE_t}{dt}$, $\dot{R}_t$ is the change of research at time t , it is the abbreviation of $\frac{dR_t}{dt}$. r_{0t} is the proportion of people engaged in labor, r_{1t} is the proportion of people engaged in education, $1 - r_{0t} - r_{1t}$ is the proportion of people engaged in research. δ_1 , δ_2 and ρ are discount factors, B_1 and B_2 are coefficient, just like A . We have the answers of r_{0t} , r_{1t} and $1 - r_{0t} - r_{1t}$:

$$r_{0t} = \frac{(\rho + \delta_1)(\rho + \delta_2)\gamma}{(\rho + \delta_1)\alpha\delta_2 + (\rho + \delta_2)\beta\delta_1 + (\rho + \delta_2)(\rho + \delta_1)\gamma}$$

$$r_{1t} = \frac{(\rho + \delta_2)\beta\delta_1}{(\rho + \delta_1)\alpha\delta_2 + (\rho + \delta_2)\beta\delta_1 + (\rho + \delta_2)(\rho + \delta_1)\gamma}$$

$$1 - r_{0t} - r_{1t} = \frac{(\rho + \delta_1)\alpha\delta_2}{(\rho + \delta_1)\alpha\delta_2 + (\rho + \delta_2)\beta\delta_1 + (\rho + \delta_2)(\rho + \delta_1)\gamma}$$

The procedure of solving the equation is indicated in appendix II.

Research is getting more and more important if new knowledge becomes more and more important, and there will be more and more people engaged in research and education.

7. DISCUSSION AND CONCLUSION

Resource per capita is becoming increasingly scarce on the earth along with the increase of population. To increase productivity, knowledge is becoming more and more important. Education aims to produce and impart knowledge [74], so education is becoming more and more important.

Religion and faith are important content of education, political institutions of a country is established on certain religion and faith, so education is mutually related with political institutions, and government wants to control education by funding schools or education policy.

A market can be full of efficiency if economic agents are at the same level in the aspect of education. A child cannot receive education if his family is poor, and he cannot find a satisfying job when he grows up, and it is not fair to him if this happens. Public education should aim to give children from poor families a chance of learning, enable all children receive education regardless of the rich or poor of their families.

Schools are competing for students, teachers and dollars, and market force is taken as a panacea for problems of the public schools [75], and market pressures have produced organizational changes in public community colleges [76]. Private schools have significant impact on the performance of public schools in states with large private school sectors, and public school students have improved education performance [67].

School education may not suit all children, home education and community schools can be established. De-schooling may be extreme, but it is very good to simplify schooling, school education need to be simplified to accommodate students. What kind of knowledge, and how many schools we need are very difficult questions, and these questions should be solved by education market. As a whole, education should be economically managed, and industrialization of education conforms to the trend of social development, certainly, education needs the support of public policy.

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APPENDIX I

$$\begin{aligned} \text{Max } & \sum_{t=0}^T e^{-\rho t} \ln(Y_t) \\ \text{s. t. } & Y_t = A E_t^\alpha r_t^\beta L_t^\beta \\ & \dot{E}_t + \delta E_t = B L_t (1 - r_t) \end{aligned}$$

To solve the equation, we need to construct a Hamilton equation:

$$H = e^{-\rho t} (\ln(A E_t^\alpha r_t^\beta L_t^\beta) + \mu_t (B L_t (1 - r_t) - \delta E_t)) \quad (1)$$

The following is the process of solving the equation

$$\begin{aligned} \frac{\partial H}{\partial r_t} &= e^{-\rho t} \left(\frac{\beta}{r_t} - \mu_t B L_t \right) = 0 \\ \Rightarrow \frac{\beta}{r_t} &= \mu_t B L_t \end{aligned} \quad (2)$$

$$\text{From } \frac{\beta}{r_t} = \mu_t B L_t, \text{ we have } \frac{r_t}{r_t} = \frac{\mu_t}{\mu_t} \quad (3)$$

$$\frac{\partial H}{\partial E_t} = e^{-\rho t} \left(\frac{\alpha}{E_t} - \mu_t \delta \right)$$

According to Euler Equation, we have

$$\begin{aligned} \frac{(e^{-\rho t} \mu_t)}{dt} &= - \frac{\partial H}{\partial E_t} \Rightarrow -\rho e^{-\rho t} \mu_t + e^{-\rho t} \frac{d\mu_t}{dt} = -e^{-\rho t} \left(\frac{\alpha}{E_t} - \mu_t \delta \right) \\ \Rightarrow -\rho \mu_t + \frac{d\mu_t}{dt} &= - \frac{\alpha}{E_t} + \mu_t \delta, \text{ or } -\rho \mu_t + \dot{\mu}_t = - \frac{\alpha}{E_t} + \mu_t \delta \\ \Rightarrow \dot{\mu}_t &= - \frac{\alpha}{E_t} + (\delta + \rho) \mu_t \end{aligned} \quad (4)$$

Replace μ_t with r_t from equation (3), and replace μ_t from equation (2), we have:

$$-\frac{r_t}{E_t} \mu_t = -\frac{\alpha}{E_t} + (\delta + \rho) \mu_t \Rightarrow -\frac{r_t}{r_t B L_t r_t} \beta = -\frac{\alpha}{E_t} + (\delta + \rho) \frac{\beta}{B L_t r_t} \Rightarrow r_t = \left(\frac{\alpha}{E_t} - (\delta + \rho) \frac{\beta}{B L_t r_t} \right) \frac{\beta r_t}{B L_t r_t}$$

We have two equations:

$$r_t = \left(\frac{\alpha}{E_t} - (\delta + \rho) \frac{\beta}{B L_t r_t} \right) \frac{\beta r_t}{B L_t r_t}$$

$$E_t + \delta E_t = B L_t (1 - r_t)$$

At an equilibrium point, we suppose $r_t = 0$, $E_t = 0$, so we have:

$$0 = \left(\frac{\alpha}{E_t} - (\delta + \rho) \frac{\beta}{B L_t r_t} \right) \frac{\beta r_t}{B L_t r_t} \quad (5)$$

$$\delta E_t = B L_t (1 - r_t) \quad (6)$$

And we have:

$$r_t = \frac{(\delta + \rho) \beta}{\alpha \delta + (\delta + \rho) \beta}$$

APPENDIX II

$$\text{Max } \sum_{t=0}^T e^{-\rho t} \ln(Y_t)$$

$$\text{s. t. } Y_t = A R_t^\alpha E_t^\beta r_{0t}^\gamma L_t^\gamma$$

$$\dot{E}_t + \delta_1 E_t = B_1 L_t r_{1t}$$

$$\dot{R}_t + \delta_2 R_t = B_2 L_t (1 - r_{0t} - r_{1t})$$

We need to construct a Hamilton equation to solve the question:

$$H = e^{-\rho t} (\ln(A R_t^\alpha E_t^\beta r_{0t}^\gamma L_t^\gamma) + \mu_{1t} (B_1 L_t r_{1t} - \delta_1 E_t) + \mu_{2t} (B_2 L_t (1 - r_{0t} - r_{1t}) - \delta_2 R_t)) \quad (1)$$

$$\frac{\partial H}{\partial r_{0t}} = e^{-\rho t} \left(\frac{\gamma}{r_{0t}} - \mu_{2t} B_2 L_t \right) = 0$$

$$\Rightarrow \frac{\gamma}{r_{0t}} = \mu_{2t} B_2 L_t \quad (2)$$

Suppose $\mu_{1t} = \frac{d\mu_{1t}}{dt}$, $\mu_{2t} = \frac{d\mu_{2t}}{dt}$, and we have:

$$\Rightarrow \frac{\gamma}{r_{0t}} = \mu_{2t} B_2 L_t \Rightarrow \frac{r_{0t}}{r_{1t}} = \frac{\mu_{2t}}{\mu_{1t}} \quad (3)$$

$$\frac{\partial H}{\partial r_{1t}} = e^{-\rho t} (\mu_{1t} B_1 L_t - \mu_{2t} B_2 L_t) = 0 \Rightarrow \mu_{1t} B_1 L_t = \mu_{2t} B_2 L_t$$

$$\Rightarrow \frac{\mu_{1t}}{\mu_{1t}} = \frac{\mu_{2t}}{\mu_{2t}} \quad (4)$$

From equation (3) and (4), we have:

$$\frac{\mu_{1t}}{\mu_{1t}} = \frac{\mu_{2t}}{\mu_{2t}} = \frac{r_{0t}}{r_{1t}} \quad (5)$$

$$-\frac{(e^{-\rho t} \mu_{1t})}{dt} = -\rho e^{-\rho t} \mu_{1t} + e^{-\rho t} \frac{d\mu_{1t}}{dt} \quad (6)$$

$$-\frac{(e^{-\rho t} \mu_{2t})}{dt} = -\rho e^{-\rho t} \mu_{2t} + e^{-\rho t} \frac{d\mu_{2t}}{dt} \quad (7)$$

$$\frac{\partial H}{\partial E_t} = e^{-\rho t} \left(\frac{\beta}{E_t} - \mu_{1t} \delta_1 \right) \quad (8)$$

$$\frac{\partial H}{\partial R_t} = e^{-\rho t} \left(\frac{\alpha}{R_t} - \mu_{2t} \delta_2 \right) \quad (9)$$

According to Euler equation, we have:

$$\begin{aligned} \frac{d}{dt} \left(e^{-\rho t} \mu_{1t} \right) &= - \frac{\partial H}{\partial E_t} \Rightarrow -\rho e^{-\rho t} \mu_{1t} + e^{-\rho t} \frac{d\mu_{1t}}{dt} = -e^{-\rho t} \left(\frac{\beta}{E_t} - \mu_{1t} \delta_1 \right) \\ \Rightarrow -\rho \mu_{1t} + \frac{d\mu_{1t}}{dt} &= - \left(\frac{\beta}{E_t} - \mu_{1t} \delta_1 \right) \Rightarrow -\rho \mu_{1t} + \frac{d\mu_{1t}}{dt} = \mu_{1t} \delta_1 - \frac{\beta}{E_t} \\ -\rho \mu_{1t} + \mu'_{1t} &= \mu_{1t} \delta_1 - \frac{\beta}{E_t} \\ \mu'_{1t} &= (\rho + \delta_1) \mu_{1t} - \frac{\beta}{E_t} \end{aligned} \quad (10)$$

According to Euler equation, we have:

$$\begin{aligned} \frac{d}{dt} \left(e^{-\rho t} \mu_{2t} \right) &= - \frac{\partial H}{\partial R_t} \Rightarrow -\rho e^{-\rho t} \mu_{2t} + e^{-\rho t} \frac{d\mu_{2t}}{dt} = -e^{-\rho t} \left(\frac{\alpha}{R_t} - \mu_{2t} \delta_2 \right) \\ \Rightarrow -\rho \mu_{2t} + \frac{d\mu_{2t}}{dt} &= - \left(\frac{\alpha}{R_t} - \mu_{2t} \delta_2 \right) \Rightarrow -\rho \mu_{2t} + \frac{d\mu_{2t}}{dt} = - \left(\frac{\alpha}{R_t} - \mu_{2t} \delta_2 \right) \\ -\rho \mu_{2t} + \mu'_{2t} &= - \left(\frac{\alpha}{R_t} - \mu_{2t} \delta_2 \right) \Rightarrow -\rho \mu_{2t} + \mu'_{2t} = \mu_{2t} \delta_2 - \frac{\alpha}{R_t} \\ \mu'_{2t} &= (\delta_2 + \rho) \mu_{2t} - \frac{\alpha}{R_t} \end{aligned} \quad (11)$$

From equation (2), (3), (4) and (5), we have:

$$\begin{aligned} \dot{\mu}_{2t} &= - \frac{r_{0t}}{r_{0t}} \mu_{2t}, \quad \dot{\mu}_{1t} = - \frac{r_{0t}}{r_{0t}} \mu_{1t} \\ \mu_{2t} &= \frac{\gamma}{B_2 L_t r_{0t}}, \quad \mu_{1t} = \frac{\gamma}{B_1 L_t r_{0t}} \end{aligned}$$

To replace μ'_{1t} , μ_{1t} in equation (10) with $\mu'_{1t} = - \frac{r_{0t}}{r_{0t}} \mu_{1t}$, $\mu_{1t} = \frac{\gamma}{B_1 L_t r_{0t}}$, and μ'_{2t} , μ_{2t} in equation (11) with $\mu'_{2t} = - \frac{r_{0t}}{r_{0t}} \mu_{2t}$, $\mu_{2t} = \frac{\gamma}{B_2 L_t r_{0t}}$, we get:

$$\begin{aligned} - \frac{r_{0t}}{r_{0t}} \frac{\gamma}{B_1 L_t r_{0t}} &\approx (\rho + \delta_1) \frac{\gamma}{B_1 L_t r_{0t}} - \frac{\beta}{E_t} \Rightarrow -\dot{r}_{0t} = ((\rho + \delta_1) \frac{\gamma}{B_1 L_t r_{0t}} - \frac{\beta}{E_t}) \frac{B_1 L_t r_{0t}}{\gamma} \\ - \frac{r_{0t}}{r_{0t}} \frac{\gamma}{B_2 L_t r_{0t}} &\approx (\delta_2 + \rho) \frac{\gamma}{B_2 L_t r_{0t}} - \frac{\alpha}{R_t} \Rightarrow -\dot{r}_{0t} = ((\rho + \delta_2) \frac{\gamma}{B_2 L_t r_{0t}} - \frac{\alpha}{R_t}) \frac{B_2 L_t r_{0t}}{\gamma} \end{aligned}$$

At an equilibrium point, $\dot{r}_{0t} = 0$, $\dot{E}_t = 0$ and $\dot{R}_t$ so we have:

$$(\rho + \delta_1) \frac{\gamma}{B_1 L_t r_{0t}} - \frac{\beta}{E_t} = 0 \Rightarrow (\rho + \delta_1) \frac{\gamma}{B_1 L_t r_{0t}} = \frac{\beta}{E_t} \quad (12)$$

$$(\rho + \delta_2) \frac{\gamma}{B_2 L_t r_{0t}} - \frac{\alpha}{R_t} = 0 \Rightarrow (\rho + \delta_2) \frac{\gamma}{B_2 L_t r_{0t}} = \frac{\alpha}{R_t} \quad (13)$$

$$\delta_1 E_t = B_1 L_t r_{1t} \quad (14)$$

$$\delta_2 R_t = B_2 L_t (1 - r_{0t} - r_{1t}) \quad (15)$$

From equation (12) and (14), we have:

$$(\rho + \delta_1) \frac{\gamma r_{1t}}{r_{0t}} \approx \beta \delta_1 \quad (16)$$

From equation (13) and (15), we have:

$$(\rho + \delta_2) \frac{\gamma(1-r_{0t}-r_{1t})}{r_{0t}} = \alpha\delta_2 \quad (17)$$

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985 From equation (16) and (17), we get:

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$$r_{0t} = \frac{(\rho + \delta_1)(\rho + \delta_2)\gamma}{(\rho + \delta_1)\alpha\delta_2 + (\rho + \delta_2)\beta\delta_1 + (\rho + \delta_2)(\rho + \delta_1)\gamma}$$

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$$r_{1t} = \frac{(\rho + \delta_2)\beta\delta_1}{(\rho + \delta_1)\alpha\delta_2 + (\rho + \delta_2)\beta\delta_1 + (\rho + \delta_2)(\rho + \delta_1)\gamma}$$

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$$1 - r_{0t} - r_{1t} = \frac{(\rho + \delta_1)\alpha\delta_2}{(\rho + \delta_1)\alpha\delta_2 + (\rho + \delta_2)\beta\delta_1 + (\rho + \delta_2)(\rho + \delta_1)\gamma}$$

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